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STATISTICAL DESIGN OF EXPERIMENTS FOR CONTINUOUS DATA

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I. <u>INTRODUCTION.</u> I wish to talk on the subject of how one would design an experiment and analyze the data when the results come in the form of a continuous curve, rather than just a single value. This is an area that would appear to have extensive application in science and engineering. For example: velocity data, trajectory data, meterological data, thrust data, etc. As I just mentioned, I am interested in the design of experiments, which means I am not concerned with the evaluation of a single curve, but many curves obtained as a result of testing under several sets of conditions, and very likely each set of conditions will have some replications.

To illustrate what I have in mind, I will use rocket motor thrust curves, although I could have used some other type of curve equally effectively. Now many results from rocket motor tests can easily be analyzed. For example: average exhaust velocity; effective (average) pressure; total impulse; specific impulse, etc. These are simple to analyze because the data for a given test usually comes in the form of one single number. However, if we want to estimate a typical or average thrust curve when a motor is tested under given conditions, this is quite a different problem.

To keep this report unclassified, the thrust data which will be discussed will be completely fictitious. The data is not, to my knowledge, appropriate for any existing rocket motor, but the general shape of the curve is similar to what may be expected for a number of motors currently in use.

The extensive information available in such areas as: Regression Analysis; Random Processes; Power Spectral Analysis; Time Series; Analysis of Covariance; Multivariate Analysis; and similar fields may easily cover the problem I am going to present. Therefore, my first
question is, if the solution is readily available in the literature, I would (1)
like some references. My second question is, if it is not readily available
but you know the answers, I will appreciate the information. (Please note (2)
that there are numbers along the margins of this report. These numbers refer to specific questions which may be found at the end of this report.)

I will conclude the introduction by stating that the five panel members, W. T. Federer, B. G. Greenberg, M. A. Schneiderman, H. L. Lucas, and H. O. Hartley have all prepared and forwarded their comments. These are included at the end of the report in the order in which they were received.

II. BASIC ASSUMPTIONS.

A. We will begin by assuming that the thrust curves to be considered in this study will look something like the one illustrated in Figure 1, although an actual curve will be somewhat more irregular than the example. The letters along the curve will represent the points which we will consider critical, and we will refer to these points many times in the future.

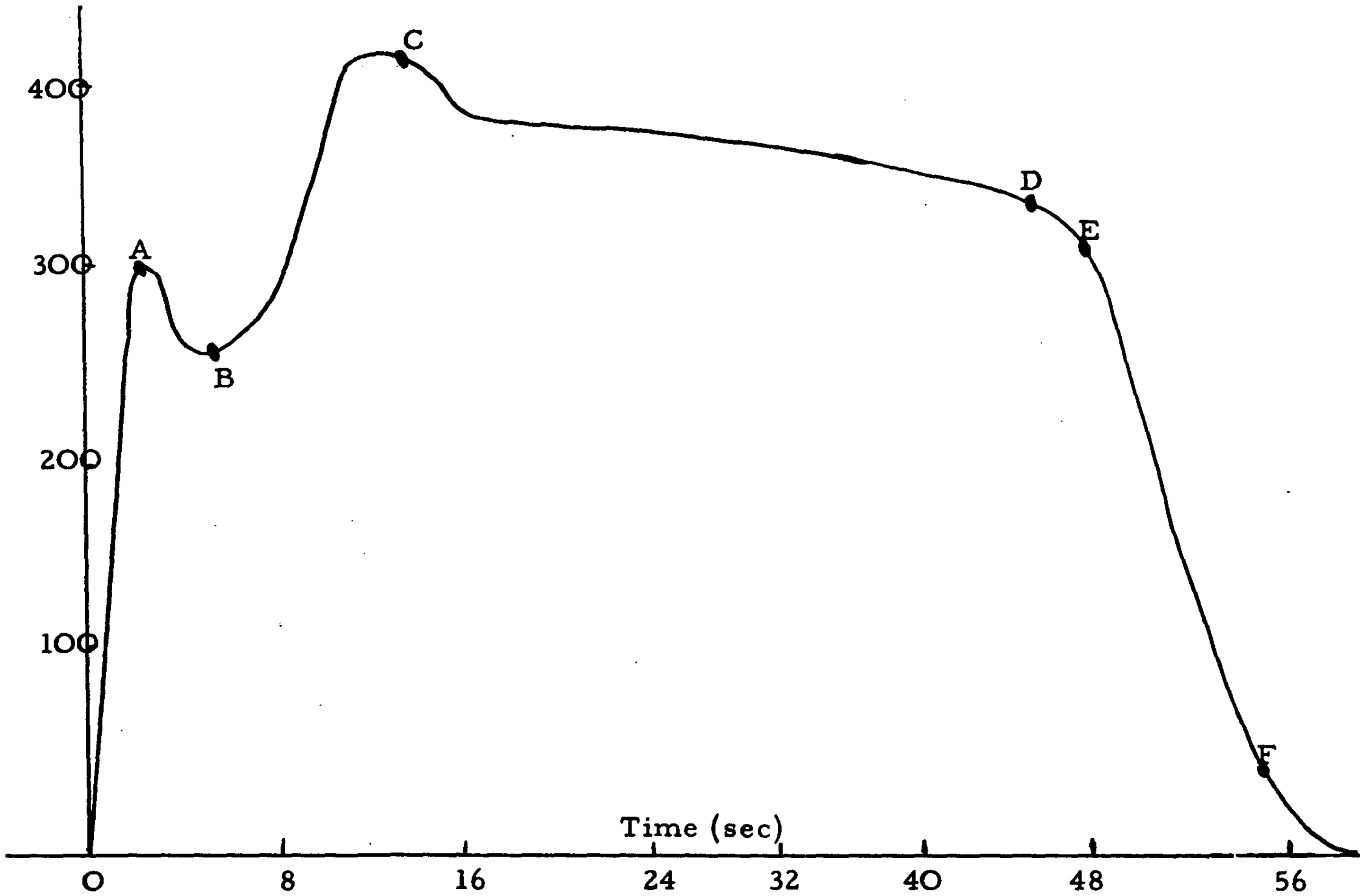


Figure 1

A Typical Thrust Curve for This Study

B. The Second assumption we will make is that the propellant mix from whence the motors are selected as well as the preconditioning temperature can influence the shape of the thrust curve. We will use motors selected from the three propellant mixes (A, B, and C), and conditioned at three temperatures (0°, 50°, and 100° F). We will use 27 motors, nine randomly selected from each mix, and of the nine, three conditioned at each of the temperatures. This arrangement is illustrated in Figure 2.

Mix	Conditioning Temperature		
	0°	50°	100°
A	3	3	3
B	3	3	3
C	3	3	3

Figure 2

Number of motors selected from each mix and conditioned at each temperature.

C. The third assumption is that temperature conditioning will have an effect similar to that illustrated in Figure 3. These three curves actually represent the average of the curves selected from Mix A and conditioned according to the specified temperature.

D. The fourth and final assumption is that the Test Engineer will want the following information:

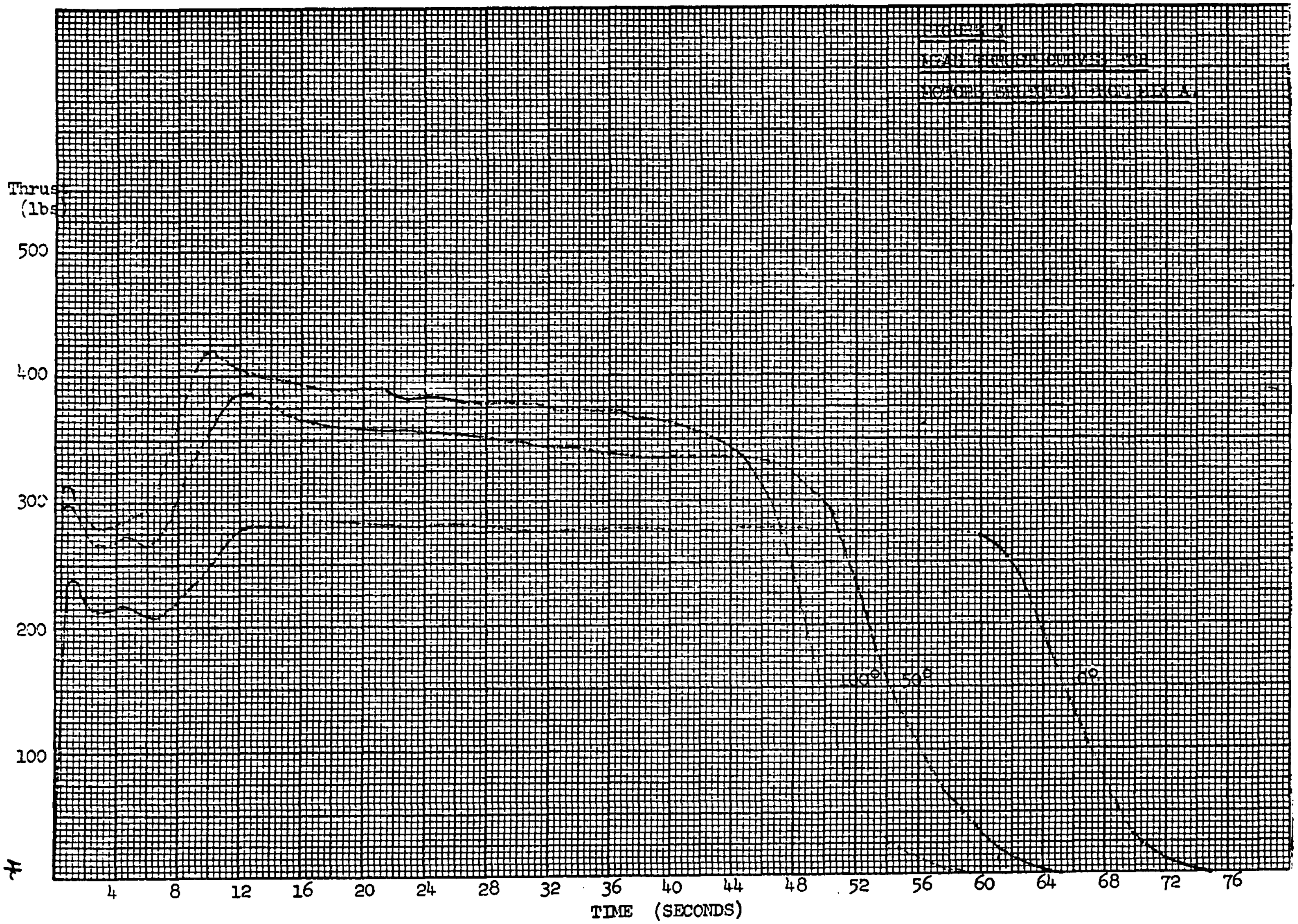
1. Does Mix difference or temperature conditioning have a significant effect upon the shape of a thrust curve?

2. For a given propellant mix or temperature what is the mean or expected thrust curve?

3. In addition to the mean thrust curve, confidence and tolerance bounds are desired.

III. <u>ANALYSIS OF VARIANCE.</u>

A. If you refer again to Figure 1, you will observe a slightly declining plateau between points C and D. I resolved, first of all, to compare


Thrust
(lbs)
500
400
300
200
100
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
TIME (SECONDS)
100°
50°
0°

performance to this plateau area. One problem is evident from Figure 2, namely the plateau areas are of different length at different temperatures. I therefore decided to study only the region from 15 to 42 seconds inclusive. For all 27 rounds, I read the values at 3 second intervals, that is to say (15, 18, ... 42 seconds). The variances appeared to be homogeneous in the region, so we performed the analysis of variance illustrated in Fig. 3.

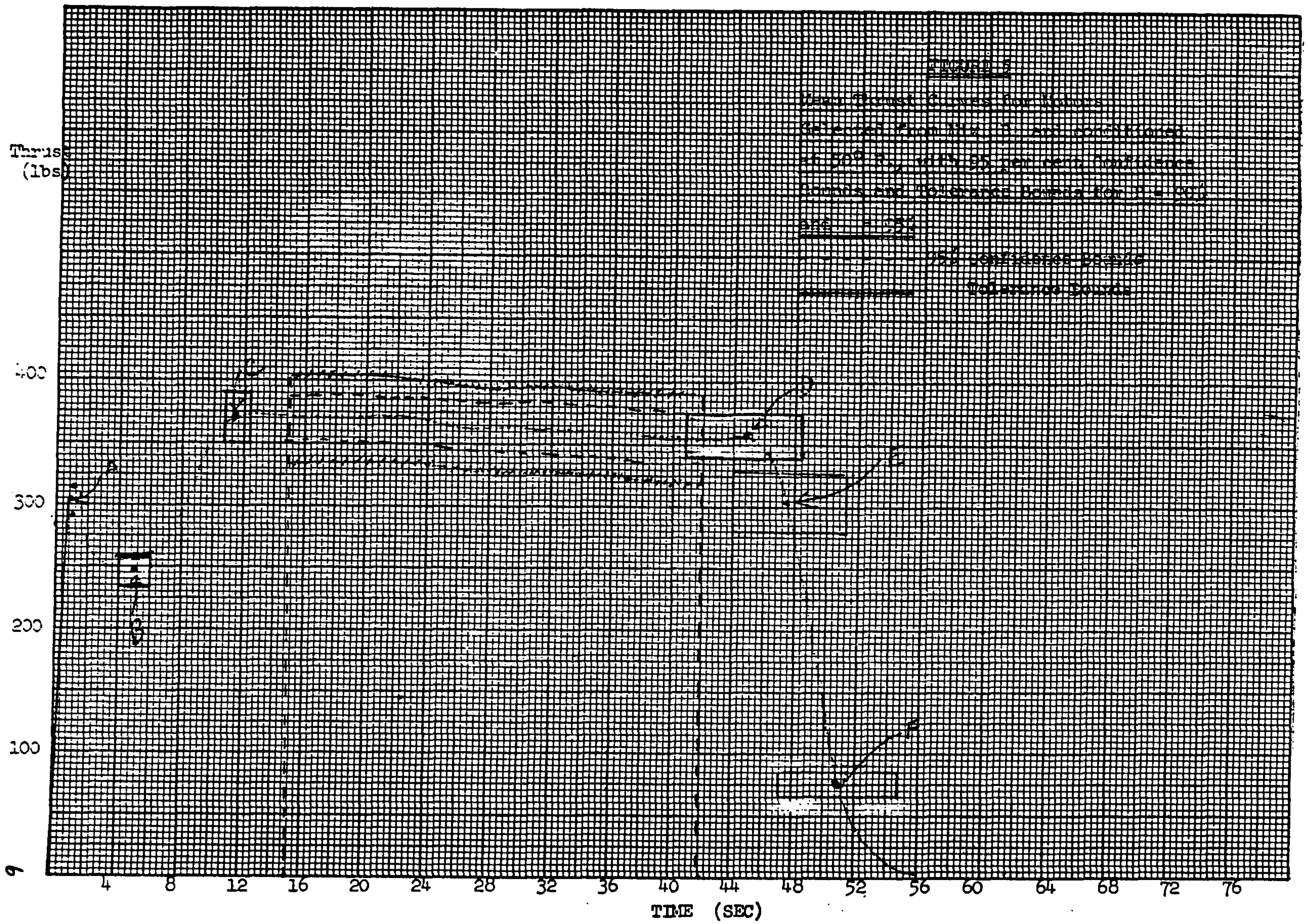
Source of Variance	d/f	SS	MS	F
Mix	2	43217.71	21608..89	82.72**
Temperature	2	488387.22	244193.61	934.78**
Time	9	12325.61	1369.50	5.24**
(Mix)(Temperature)	4	19794.61	4948.65	18.94**
(Mix)(Time)	18	513.70	28.52	0.11
(Time)(Temperature)	18	2954.70	164.15	0.63
(Mix)(Time)(Temperature)	36	24459.88	679.44	2.60**
Error	180	47022.04	261.23 $=(16.16)^2$	
Total	269	638674.97		

*Significant at .05 level (Single and double asterisk used in later tables
**Significant at .01 level have the same meaning indicated here.)

Figure 4

Analysis of Variance of Thrust Values between 15 and 42 seconds for 27 motors

From Figure 4 it appears that for these curves, temperature has a highly significant effect, mix has an important effect, and there is a significant, downward trend during this time interval. While the mix-temperature interaction is significant, its F value is relatively small; so we will assume it is not really critical. It may also be observed that the pooled estimate of variance is 261.23, and we will therefore assume a standard error of 16 lbs. with 180 degrees of freedom. I then proceeded to construct 9 graphs, one for each combination of temperature and mix. Figure 5 illustrates the graph for Mix B and 50° temperature. I took the sample of three and plotted the mean values for the interval from 15 to 42 seconds. Next, I used the pooled estimate of variance and plotted 95% confidence bounds on this curve, and then on the outside of this, I plotted tolerance bounds, $\gamma = 95\%$, $P = 90\%$.


Mean Thrust Curves for Motors
95% Confidence Bounds
Tolerance Bounds
Thrust (lbs)
400
300
200
100
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
TIME (SEC)
A
B
C
D
E
F

Clearly, the question at this point is that of the propriety of arbitrarily taking a time curve, observing the values at stated intervals (every three seconds in this case) and considering time, along with mix and temperature, (3) as one of the treatments in the analysis variance. It is interesting that the error term has 180 degrees of freedom. Had we arbitrarily chosen 2 second intervals, for example, instead of 3 second intervals for taking our readings the degrees of freedom for error would have increased to 270. There is clearly something illogical at this point.

B. Referring again to Figure 1, you will note that I have arbitrarily selected six critical points. I then proceeded by performing an analysis of variance for both the X and Y component for each of these critical components. The Analysis of Variance for the Y component of A is given in Fig. 6 and the X component in Figure 7.

Sources of Var	SS	d/f	MS	F
Mix	198	2	99	0.74
Temp	27,746	2	13,873	104.30
Interaction	200	4	50	0.38
Error	2,398	18	133	
Total	30,542	26		

Figure 6

Analysis of Variance for the Y (Thrust in lbs) Component at Point A.

Sources of Var	SS	d/f	MS	F
Mix	.1267	2	.0634	1.80
Temp	.8339	2	.4170	11.88**
Interaction	.3129	4	.1564	4.48*
Error	.6317	18	.0351	
Totals	1.9052	26		

Figure 7

Analysis of Variance for the X (Time in seconds) Component for Point A

From this it may be observed that temperature had a significant effect but mix did not. Returning to Figure 5, the mean value was located for Point A and a confidence rectangle was drawn about it. This procedure was also followed for the other five critical points and these points were then connected. For the time component, temperature had a significant effect for all six critical points, mix at points D, E, and F. For the thrust Component, temperature had a significant effect at all points except F and mix had a significant effect at points B, D, and E.

One will obviously be concerned at this point by the fact that the six critical points were arbitrarily chosen and are not precisely defined. This could easily result in considerable inaccuracy in collecting data for these points. However, this fact will not necessarily be emphasized since it is not really relevant to the basic purpose of this paper.

However, the matter of performing separate analysis of the time and thrust components of each critical point is highly questionable, and I am certain that a procedure applying the bivariate normal distribution would be in order.

Referring either to Figures 1 or 5, it would appear reasonable that if an analysis of variance is appropriate for C-D, then it would probably be equally appropriate for A-C and possibly for E-F. In fact, it would appear more sensible than attempting to locate and evaluate critical points.

IV. REGRESSION ANALYSIS. A second approach, and one which I feel offers more promise is in the area of regression analysis and polynomial fitting. I will discuss a few ideas along this line at the present time.

If you will refer to Figure 1 again, I arbitrarily broke the graph up into four distinct segments. These are: O-A; A-C; C-D; and E-F. Then, for all 27 motors, I fitted the most appropriate polynomial, that is to say, I fitted a cubic to A-C and straight lines to the other three segments.

I will discuss the procedures I followed in analyzing segment C-D and state little more than that analyses were performed on the other segments, and upon completion all segments were plotted until they intersected. Using the values from 15 to 45 seconds and recording the data at 3 second intervals, a straight line was fitted for all 27 sets of data and an analysis of variance was performed for a, b, and r in the equation $Y = a + b. (X-30)$. Figure 8 gives the analysis of variance and mean values for a, while Figure 9 gives the analysis of variance and means values for b, (note that mix had no significant effect upon b, so the mean values reflect only temperature). (6) Neither mix nor temperature had any significant effect upon r (the correlation coefficient), but the mean value of the 27 correlation coefficients was 82%.

Sources of Variation	SS	d/f	MS	F
Mix	4411	2	2206	6.28**
Temperature	45959	2	22980	65.42**
Interaction	1959	4	488	1.39
Error	6323	18	351	
Total	58644	26		

Mix	Temperature		
	0°	50°	100°
A	280	348	375
B	292	365	385
C	264	309	376
Ave	278	340	379

Figure 8

The Analysis of Variance and the Mean Values for a, when Fitting the Equation, $Y = a + b \cdot (X-30)$ $15 \text{ sec} \leq x \leq 45 \text{ sec}$ (all means computed from a sample of 3, y = thrust in lbs., x = time in seconds).

Sources of Variation	SS	d/f	MS	F
Mix	O. 618	2	O. 319	1. O4O
Temperature	3. 689	2	1. 844	6. 2O9**
Interaction	O. 967	4	O. 242	. 815
Error	5. 34O	18	O. 297	
Total	1O. 644	26		

Temperature	0°	50°	100°
Mean Value b	-. 41O	-. 881	-1.315

Figure 9

The Analysis of Variance and Mean Values for b,
obtained from fitting the equation $Y = a + b\cdot(X-30)$,
15 sec. $\leq x \leq$ 45 sec. (all means computed from a sample of 9)

The data in Figure 9 indicates that "b" increases almost linearily with temperature. In fact, the formula $b = -.410 - (.00905)$ temp, might serve as a guide for selecting "b" in the region $0^{\circ} \leq \text{temp} \leq 100^{\circ}$. If one desires a formula for estimating "a" in the region for $0^{\circ} \leq \text{temp} \leq 100^{\circ}$, he might try the formula: $a = 278 + 1.4 \cdot (\text{temp}) - .0041(\text{temp})^2$ which averages out the mix effect.

Again referring to Figures 8 and 9, one may observe that the standard error for "a" is 18. 7 lbs. with 18 degrees of freedom, and the standard error for "b" is O. 545, also with 18 degrees of freedom.

In addition to this, each time a line is fitted by least squares, it is possible to obtain a standard error of estimates and standard errors for "a" and "b". For the 27 curves, I pooled these standard errors and obtain the following results:

Pooled Standard Errors of estimates:
3. 4O lbs. with 243 d/f.

Pooled Standard Error for "b":
O. 324 with 243 d/f.

Pooled Standard Error for "a":
1.025 with 243 d/f.

As may be expected, the estimates for the standard error for both "a" and "b" are larger in Figures 8 and 9 than the pooled estimates listed above. This is reasonable since the estimates in Figures 8 and 9 include the dispersions that exist among curves from the same lot and conditioned at the same temperature, while the pooled estimates reflect the variation within only a single curve.

Inasmuch as I am attempting to classify any curves which come from a given mix and a given temperature, it would seem more appropriate to use the estimates of variability in Figures 8 and 9. One other argument for this lies in the fact that when the standard error of estimates was computed for each of the 27 curves, it was computed from 11 points, selected from the thrust curves at 3 second intervals. 15 sec $\leq$ time $\leq$ 45 sec. Again the question arises concerning the arbitrariness in choosing 3 second intervals instead of some other intervals.

Now confidence bounds for a regression line at a point x_i may be computed from the formula

$$\bar{Y}_i - t \cdot S(\bar{Y}_i) \leq E(Y_i) \leq \bar{Y}_i + t \cdot S(\bar{Y}_i)$$

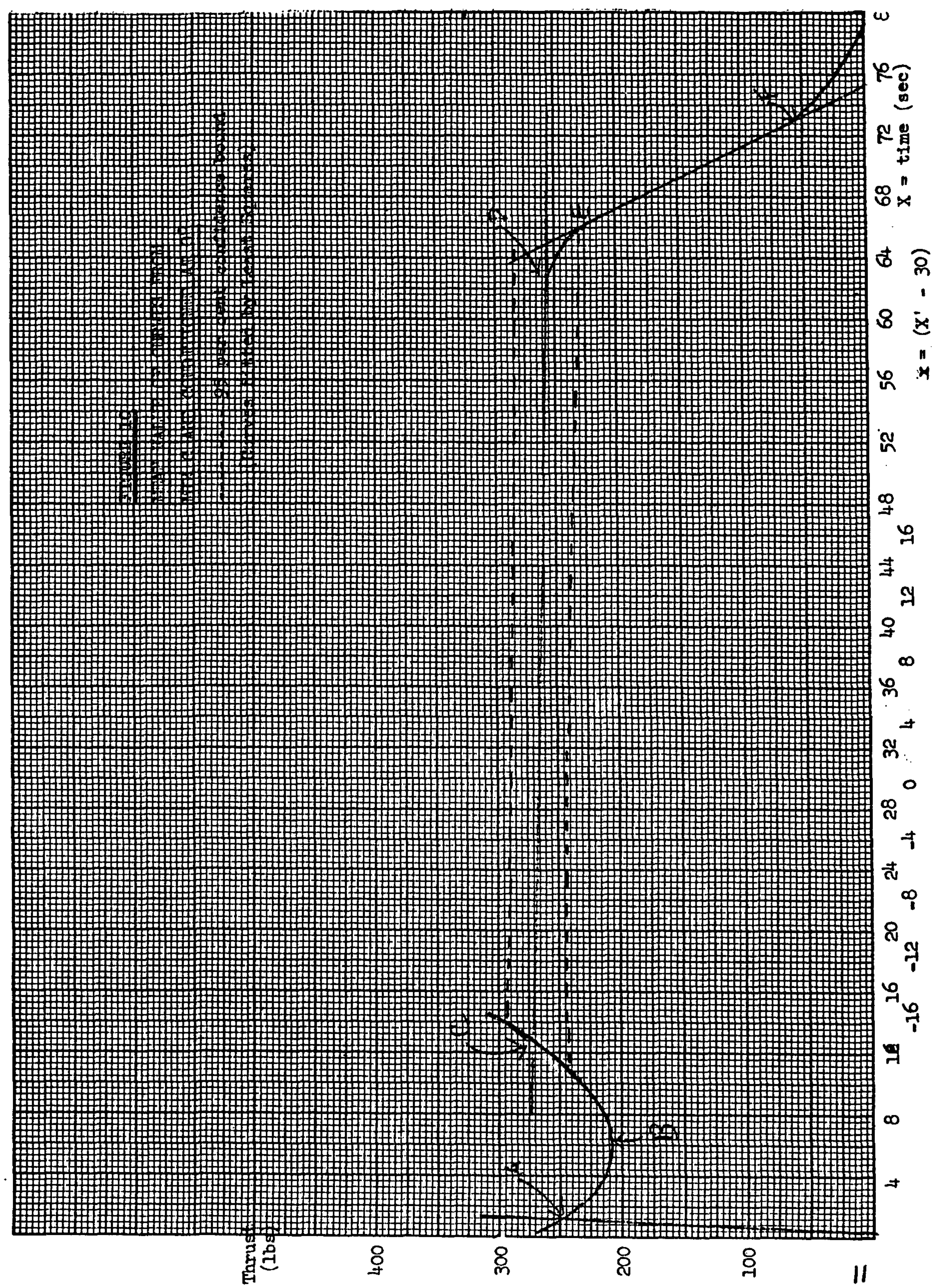
where $\bar{Y}_i = a + b\,x_i$

and $S^2(\bar{Y}_i) = S^2(a) + x_i^2 \cdot S^2(b); \; \bar{x}_i = 0.$

Since each "a" represents an average of 3 numbers and each "b" represents an average of 9 numbers, we have:

$$S^2(\bar{Y}_i) = \frac{351}{3} + x_i^2 \cdot \frac{.297}{9} = 117 + .033\,x_i^2 \,.$$

These ideas are illustrated in Figure 10, in which we consider Mix C and a temperature of 0°, and fit the curve from 15 sec $\leq X \leq$ 45 sec. (Segment C-D). This curve is given by $\bar{Y} = 264 - .410\,X$. or $\bar{Y} = 264 - .410 \cdot (X - 30)$,


Thrust (lbs)
400
300
200
100
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
80
X = time (sec)
-16
-12
-8
-4
0
4
8
12
16
x = (X' - 30)
A
B
C
D
E
F

and the variance $S^2(\bar{Y}) = 117 + .033\, X_i^2$ was used to compute a confidence bound about the curve. Incidentally, the confidence bound in Figure 10 is very close to the one illustrated in Figure 5.

The procedures for fitting the segments (O-A) and (E-F) could be quite similar to that of fitting the segment (C-D). In fact the segment (O-A) should be even simpler. To fit the segment (A-C), it is suggested that a cubic equation be fitted, using data points at one second intervals. It is further suggested that orthogonal polynomials be used when fitting a cubic or higher degree equation to simplify the process of obtaining the variance[(7)] and confidence bounds.

<u>CONCLUSIONS.</u> Frequently it is desired to design an experiment when the results of the test are a continuous curve rather than a single quantitative value. Scientists and Engineers frequently want to know whether certain levels of a given treatment will have a significant effect upon the curve obtained, and what will an average or expected curve be for a given set of conditions.

I have made a few suggestions based largely upon analysis of variance or regression analysis. I will greatly appreciate comments on the proposed solutions, but more important, I would like suggestions for better approaches to the problem.

<u>QUESTIONS.</u>

1. Has the problem of designing an experiment when the results come as a continuous curve rather than a single value ever been solved? If so, are useful references available?

2. Do you have any ideas of additional approaches beside those suggested in the paper?

3. When studying a section of the curve such as C-D, is there any justification in arbitrarily selecting a set of times between C and D, computing the thrust at each of these times for all available curvex, and performing an analysis of variance similar to that given by Figure 4? Perhaps this would be in order with certain changes in procedure.

4. What procedures would you suggest when attempting to locate a point in terms of both its X and Y components and then obtaining both a confidence and tolerance region about this point?

5. Is the analysis of variance for a, b, and r, as illustrated in Figures 8 and 9 appropriate?

6. Have you any suggestions regarding the validity of the techniques, using regression analysis, that were discussed in this section?